ON 1-BOUNDING MONOTONIC TRANSFORMATIONS WHICH ARE EQUIVALENT TO HOMEOMORPHISMS

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN THE
UNIVERSITY OF MICHIGAN

BY

EDITH R. SCHNECKENBURGER

Reprinted from THE AMERICAN JOURNAL OF MATHEMATICS,
Vol. LXIII, number 4, pages 768 to 784.

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ON 1-BOUNDING MONOTONIC TRANSFORMATIONS WHICH ARE EQUIVALENT TO HOMEOMORPHISMS.* [1]

By EDITH R. SCHNECKENBURGER.

1. Introduction. A monotonic [2] transformation of a space A on to a space B is a transformation such that the inverse of each point b of B is connected; in other words, the 0-dimensional Betti number of the inverse of b is zero. In this paper such a transformation will frequently be called an R_0^0-transformation. It could also be called a 0-bounding transformation.

A 1-bounding transformation, or R_0^1-transformation, of a space A on to a space B is a transformation such that for each point b of B the 1-dimensional Betti number of the inverse of b is zero.

An R_0^{01}-transformation is a transformation which is monotonic and 1-bounding. In general, an $R_0^{01\cdots n}$-transformation of a space A on to a space B is a transformation such that for every point b of B the i-dimensional Betti number, $i = 0, \cdots, n$, of the inverse of b is zero.

In this paper we consider a single-valued, continuous, R_0^{01}-transformation T of a space A on to a space B such that T is equivalent to a homeomorphism. We shall say that a transformation $T(A) = B$ is equivalent to a homeomorphism when B is homeomorphic to A.

In a paper [3] by Roberts and Steenrod it is shown that a single-valued, continuous, R_0^{01}-transformation of M, a compact two-manifold without boundary, gives a space homeomorphic to M. This is a generalization of a theorem of R. L. Moore which states this result for the case in which M is a two-sphere.[4] These results suggest the consideration of the following questions:

1. What spaces, other than two-manifolds, satisfy the condition that every T of the type under consideration is equivalent to a homeomorphism?

2. What conditions must be placed upon a particular T to make it equivalent to a homeomorphism on a given space A?

In section 3 we state conclusions of the nature described in question 2. In considering question 1, we have confined our attention to the study of a

* Received July 29, 1940; revised March 4, 1941.

[1] Presented to the American Mathematical Society, September 7, 1939.

[2] Defined as a monotone transformation by C. B. Morrey, *American Journal of Mathematics*, vol. 57 (1935), p. 18.

[3] J. H. Roberts and N. E. Steenrod, *Annals of Mathematics*, vol. 39 (1938), Theorem 1, p. 854.

[4] R. L. Moore, *Transactions of the American Mathematical Society*, vol. 27 (1925), Theorem 25, p. 247.

space A which is a Peanian continuum imbedded in a two-manifold M. Our results are:

(a) If A is cyclic, A is a simple closed curve or the two-manifold M.

(b) If A is acyclic, A is an arc. (This is true for any acyclic Peanian continuum.)

(c) If A is neither cyclic nor acyclic and certain restrictions are put upon the cut points of A, then A is a finite or countably infinite set of simple closed curves tangent to each other at a single point p.

We have concluded that a complete characterization of A, if possible, would be too lengthy and tedious to be interesting.

The author wishes to thank Professor W. L. Ayres of the University of Michigan who suggested the investigation of this problem and gave many helpful criticisms and suggestions during the preparation of this paper.

2. Notation. In this paper, the following notation is used: $H \equiv$ homeomorphism; $T \equiv$ single-valued, continuous, R_0^{01}-transformation; $X \equiv$ Peanian continuum contained in a compact metric space; $M \equiv$ compact two-manifold without boundary; $A \equiv X$ imbedded in M; $B \equiv T(X)$ where B is contained in a metric space, and it is assumed that B is never a single point.

We shall also have occasion to use these symbols: $R^i(S) \equiv \bmod 2$, i-dimensional Betti number of a set S; $\langle xy\rangle \equiv xy - x - y$ where xy is an arc with endpoints x and y; $xy\rangle \equiv$ arc xy minus end point y; $\langle xy \equiv$ arc xy minus end point x.

A knowledge of the cyclic element theory, as developed in a paper by Kuratowski and Whyburn,[5] will be presupposed.

3. Conditions that a particular transformation be equivalent to a homeomorphism. A vertex y of a finite graph X will be called *non-essential* if it is the end point of exactly two 1-cells xy and yz and $x \neq z$; otherwise, y is an essential vertex.

LEMMA 1. *The 1-cells of any finite graph X may be so re-defined that every vertex is essential.*

THEOREM 1. *If X is a finite connected graph, a necessary and sufficient condition that $B = T(X)$ be homeomorphic to X is that $T(u) \neq T(y)$ for any two essential vertices u and y of order $k \neq 2$.*

COROLLARY. *If X is a simple closed curve, $B = T(X)$ is also a simple closed curve.*[6]

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[5] C. Kuratowski and G. T. Whyburn, *Fundamenta Mathematicae*, vol. 16 (1930), pp. 305-331.

[6] G. T. Whyburn, *Bulletin of the American Mathematical Society*, vol. 42 (1936), p. 66.

LEMMA 2. *If H_0 is an order preserving homeomorphism between a closed subset S of an arc $\alpha = a_1a_2$ and a subset $H_0(S)$ of an arc $\beta = b_1b_2$ and $a_1 + a_2 \subset S$, $b_1 + b_2 \subset H_0(S)$, then there exists a homeomorphism H between α and β which agrees with H_0 on S.*

THEOREM 2. *If X is a tree, S is the set of all end points and branch points of X and T is a single-valued, continuous, R_0^0-transformation such that $T(u) \neq T(y)$ where u and y are any two points of S, then $B = T(X)$ is a tree which is homeomorphic to X.*

The proofs of the lemmas and theorems, stated above, are not included in this paper since the method of proof in each case is quite obvious. Lemma 1 is used in the proof of Theorem 1 and Lemma 2 in the proof of Theorem 2.

4. Conditions that every transformation be equivalent to a homeomorphism.

LEMMA 3. *If A is a proper subset of M which is cyclicly connected but is not a simple closed curve, there is an acyclic continuum in A which separates A.*

Let x be any point of A which is on the boundary of some complementary domain of A in M. Choose an open 2-cell neighborhood E of x in M such that: (1) $A \cdot (M - \bar{E}) \neq 0$, and (2) $A \cdot (M - \bar{E}) \neq \langle xy \rangle$, where $x + y \subset \bar{E}$. Since A is not a simple closed curve, it is always possible to select an E satisfying condition (2). Denote by C the simple closed curve which is the boundary of E and by N the component of $A - A \cdot C$ which contains x. Then $\bar{N}$ is a continuous curve [7] which is contained in $E + C$.

It follows from a theorem of W. L. Ayres [8] that a point q of $C \cdot \bar{N}$ is separated from x in $\bar{N}$ by a finite number n of arcs [9] α_i of $\bar{N}$, where n is the number of components of $\bar{N} - x - q$ which have x and q as limit points. Let N_q be the component of $\bar{N} - \sum_{i=1}^{n} \alpha_i$ which contains q. The set of components N_q for all points of $C \cdot \bar{N}$ is a collection of open sets covering the closed and compact set $C \cdot \bar{N}$. By the Heine-Borel theorem a finite number of these components cover $C \cdot \bar{N}$. Denote these components by N_i.

For each N_i there is a finite number of arcs α_{ji} such that $\sum_j \alpha_{ji} = W_i$ separates x from $C \cdot N_i$ in $\bar{N}$. If, for some $i \neq k$, $\alpha_{ji} \cdot \alpha_{hk} \neq 0$ and the set

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[7] R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, theorem 57, p. 232.

[8] W. L. Ayres, *Proceedings of the National Academy of Sciences*, vol. 14 (1928), theorem 2, p. 205.

[9] An arc may be a single point.

$\alpha_{ji} + \alpha_{hk} = V$ is not acyclic, we replace V by a subcontinuum V' which is acyclic. The set V' is obtained by removing from each simple closed curve of V the interior points of a subarc which contains no branch points. There is such an arc in each simple closed curve of V because the number of arcs α_{hk} is finite and hence there can be at most a denumerable number of branch points of V on any simple closed curve contained in V. It is evident that V' is connected, closed, and acyclic.

We repeat the above process for every pair of arcs of $W = \sum_i W_i$ whose sum contains a simple closed curve. Thus, we obtain a finite set of acyclic continua β_i, $i = 1, \cdots, s$, $\beta_i \cdot \beta_j = 0$, $i \neq j$, where β_i is either an arc α_{hk} or an acyclic continuum contained in the sum of two or more of these arcs.

Then $\bar{N} - \sum_{i=1}^{s} \beta_i = R + Q$, $\bar{R} \cdot Q = R \cdot \bar{Q} = 0$, R connected, $x \in R$ and $C \cdot \bar{N} \subset Q$. Let $x_1 b_1$ be an arc of $R + \sum_{i=1}^{s} \beta_i$ such that $x_1 b_1 \rangle \subset R$, $b_1 \in \beta_1$. In general, denote by $x_i b_i$, $i = 2, \cdots, s$, an arc such that $\langle x_i b_i \rangle \subset R - \sum_{j=1}^{i-1} x_j b_j$, x_i a point of $\sum_{j=1}^{i-1} x_j b_j$, and b_i a point of β_i. Let $A_1 = \sum_{i=1}^{s} (\beta_i + x_i b_i)$. The set A_1 is closed because it consists of a finite number of acyclic continua. Since each $x_i b_i$ has one and only one point in $\sum_{j=1}^{i-1} (\beta_j + x_j b_j)$, A_1 is connected and acyclic.

There are two cases to be considered. Let us suppose first that $R + \sum_{i=1}^{s} \beta_i \not\subset A_1$. Then, $\bar{N} - A_1 = S_1 + S_2$, $\bar{S}_1 \cdot S_2 = S_1 \cdot \bar{S}_2 = 0$, $R - A_1 \cdot R \subset S_1$, $C \cdot \bar{N} \subset S_2$. Since any arc joining $R - A_1 \cdot R$ to $A \cdot (M - \bar{E})$ must contain a point of $C \cdot \bar{N}$, the set A_1 is an acyclic continuum separating A.

In the second case, $A_1 = R + \sum_{i=1}^{s} \beta_i$. Let z be a point of order 2 of R, $z_1 z z_2$ any subarc of R containing no branch point of A_1, and A'_1 and A''_1 the components of $A_1 - z_1 z z_2$. Since A is cyclicly connected, there is an arc ρ which joins A'_1 and A''_1 and does not contain z. This arc ρ contains a subarc ρ_1 having only its endpoints in A'_1 and A''_1. These endpoints must be contained in $\sum_{i=1}^{s} \beta_i$, hence, $\rho_1 \cdot (z_1 z_2) = 0$. Let $A_2 = A_1 - \langle z_1 z_2 \rangle + \rho_1$. Then $A - A_2 - z_1 z_2 \neq 0$ because it follows from condition (2) that $A - A \cdot \bar{E}$ cannot be a subarc of ρ_1. Hence, the acyclic continuum A_2 separates $\langle z_1 z_2 \rangle$ from points of $A - A_2 - z_1 z_2$. Thus, in either case, there is an acyclic continuum in A which separates A.

THEOREM 3. *If A is cyclicly connected and $B = T(A)$ is homeomorphic to A for every T, either $A \equiv M$ or A is a simple closed curve.*

The fact that A can be the two-manifold M follows from a theorem of Roberts and Steenrod.[3] Hence, we need consider only the case in which A is a proper subset of M.

We suppose that A is not a simple closed curve. By Lemma 3 there is an acyclic continuum A_1 in A which separates A. Define $T(A)$ as follows: $T(A_1)$ is a single point, T is a homeomorphism on $A - A_1$. Then $T(A_1)$ is a cut point of B and B is not homeomorphic to A because A is cyclicly connected.

Hence, if there exists any set A which is a proper subset of M and satisfies the hypothesis of the theorem, A must be a simple closed curve. The existence of A follows from the Corollary to Theorem 1.

THEOREM 4. *If X is acyclic and $B = T(X)$ is homeomorphic to X for every T, then X is an arc.*

Let α_1 be any arc of X joining two end points of X, $\sum_i r_i$ the set of branch points of X which are contained in α_1, and S_i the sum of all components D_{ij} of $X - r_i$ such that $D_{ij} \cdot \alpha_1 = 0$. Define $T(X)$ as follows: T is a homeomorphism on α_1, $T(S_i) = T(r_i)$. Then T is a single-valued, continuous, R_0^{01}-transformation. Further, since $T(X) = T(\alpha_1) = B$ and T is a homeomorphism on α_1, B is an arc. Then B cannot be homeomorphic to X unless X is also an arc. Thus $S_i = 0$ for every i. Since every transform of an arc is an arc when T is monotonic,[6] B is homeomorphic to X for every T when X is an arc. Hence, if X is acyclic, it is an arc.

THEOREM 5. *If $B = T(X)$ is homeomorphic to X for every T, then the closed extension of every component of $X - C_k$ contains at least one C_i, where C_j is a true cyclic element of X.*

Let x_j be any cut point of X which is contained in some C_k and let D_{jh} be any component of $X - C_k$ which has x_j as limit point. Denote by D_j the sum of all the components D_{jh} such that $\bar{D}_{jh}$ contains no C_i. We shall prove that D_j is vacuous. Define a transformation $T(X) = B$ as follows: $T(D_j + x_j) = x_j$, $T(x) = x$ where x is any point of $X - \sum_j D_j$.

It can be shown that T is a single-valued, continuous, R_0^{01}-transformation. Obviously, T is single-valued. It is also evident that T is continuous over any set of points contained in one set D_j and over $X - \sum_j D_j$. Hence, to prove that T is continuous it is necessary to consider only sequences of points $\{d_j\}$, where d_j is a point of D_j. Let d be the sequential limit point of such a sequence d_{j_i}. Then d is contained in $X - \sum_j D_j$ because each set D_k is open and can contain no limit point of $X - D_j$. Therefore, $T(d) = d$. Since for every $\epsilon > 0$ there is a $\delta > 0$ such that $S(d, \delta)$ is contained in a connected

subset of $S(d, \epsilon)$ and since the cut point x_{j_i} separates d and d_{j_i}, it follows that all but a finite number of x_{j_i} are contained in $S(d, \epsilon)$; that is, d is a sequential limit point of $\{x_{j_i}\}$. Hence, $T(d) = d$ is a limit point of $\{T(x_{j_i})\} = \{x_{j_i}\} = \{T(d_{j_i})\}$. This proves that T is continuous.

The continuum $D_j + x_j$ contains only cyclic elements, namely, cut points and end points of X, and is therefore a continuous curve.[10] From this fact and the fact that $D_j + x_j$ is defined so that it contains no simple closed curve, it follows that $R^1(D_j + x_j) = 0$. Thus, T satisfies all the required conditions.

Since T is the identity transformation on each C_i, $T(C_i)$ is a true cyclic element of B. Furthermore, B_i, any true cyclic element of B, is contained in $T(C_j)$ for some j.[11] Hence, $B_i \equiv T(C_j)$. Let b_k be any cut point of B which is contained in some B_k. The closed extension of every component of $B - b_k$ contains at least one $T(C_j) = B_i$, because T is homeomorphic on $X - \sum_j D_j$. Thus, B is not homeomorphic to X unless D_{jh} of X for every j and h satisfies the condition that $\bar{D}_{jh}$ contains at least one C_i.

LEMMA 4. *If S is a closed subset of X and each component S_i of S is contained in a continuum L_i of X where $R^1(L_i) = 0$ and $L_i \cdot L_j = 0$, $i \neq j$, then S is contained in a continuum W of X such that $R^1(W) = 0$.*

Let $L = \sum_i L_i$ and define $T(X) = B$ as follows: $T(L_i)$ is a single point, T is a homeomorphism on $X - L$. We shall see that this transformation satisfies all the conditions required of T. It is evident from the definition that T is single-valued. Also, T is an R_0^{01}-transformation because $T^{-1}(b)$ for every point b of B is a single point or some L_i, L_i is a continuum, and $R^1(L_i) = 0$. We prove that T is continuous by showing that the collection of all the sets $T^{-1}(b)$ is an upper semi-continuous collection of continua filling X. Since every set $T^{-1}(b)$ which is in L is a component L_i of L and each L_i is compact, it follows from a theorem of R. L. Moore [12] that the sets $T^{-1}(b)$ which are in L form an upper semi-continuous collection of continua filling L. Furthermore, if $\{x_i\} = \{T^{-1}(b_i)\}$ is a sequence of points of $X - L$ which converges to x, a point of $T^{-1}(b)$, then every infinite subsequence of $\{T^{-1}(b_i)\}$ converges to x. Hence, all the sets $T^{-1}(b)$ form an upper semi-continuous collection of continua filling X. We conclude that T is continuous, for Kuratowski [13] has shown that an upper semi-continuous collection of closed sets

[10] G. T. Whyburn, *American Journal of Mathematics*, vol. 50 (1928), theorem 10, p. 177.

[11] G. T. Whyburn, *American Journal of Mathematics*, vol. 56 (1934), theorem (3.1), p. 298.

[12] See footnote 7, Theorem 20, p. 340.

[13] C. Kuratowski, *Fundamenta Mathematicae*, vol. 11 (1928), theorems 2, 3, pp. 172-173.

filling a compact metric space is equivalent to a single-valued, continuous transformation defined on the space.

It is readily seen that $T(L)$ is a Moore-Kline subset of B.[14] The set $T(L)$ is closed and compact because T is continuous. Since T is also a $R_0{}^0$-transformation, connectedness is an invariant of T^{-1} [15] and, hence, each component of $T(L)$ is a single point. It follows from a theorem of Zippin [16] that $T(L)$ is contained in a tree B_1 of B.

Since T is continuous and monotonic, $T^{-1}(B_1)$ is a continuum. It has been shown by Vietoris [17] that the one-dimensional Betti number of a compact closed set in a metric space is an invariant of T. Therefore, $R^1[T^{-1}(B_1)] = 0$ because $R^1(B_1) = 0$. Since $T(L) \subset B_1$, $L \subset T^{-1}(B_1)$. Hence, $S \subset T^{-1}(B_1)$ and $T^{-1}(B_1)$ is the set W described in the statement of the theorem.

THEOREM 6. *If $B = T(A)$ is homeomorphic to A for every T and every component of $\bar{P} \cdot C_i$ is contained in a continuum L_{ij} of C_i such that $R^1(L_{ij}) = 0$, where P is the set of all cut points A and C_i is any true cyclic element of A, then $A \equiv \sum_i C_i$, C_i is a simple closed curve, and $C_i \cdot C_j = p$, a single point, for $i \neq j$.*

It follows from Lemma 3 that the continua L_{ij} of C_i are contained in a continuum L_i of C_i such that $R^1(L_i) = 0$. We shall prove first that the components $\bar{S}_j$ of $\overline{\sum_i L_i}$ also satisfy the hypothesis of Lemma 3. Let S_1 be any component of $\sum_i L_i$. Then $S_1 = \sum_n L'_n$ where $L'_n \subset C'_n$. If $S_1 = \sum_{n=1}^{k} L'_n$, where k is finite, $S_1 \equiv \bar{S}_1$. Also, $N_1 = \sum_{n=1}^{k} C'_n$ is a Peanian continuum. If $S_1 = \sum_{n=1}^{\infty} L'_n$, then $N_1 = \sum_{n=1}^{\infty} C'_n$ is a connected subset of A which contains only cyclic elements of A. Hence, by a theorem of Whyburn,[18] $\bar{N}_1$ is a continuous curve and each point of $\bar{N}_1 - N_1$ is a cyclic element of A, namely, an end point or a cut point of A. Since $\bar{N}_1$ is a continuous curve in both cases, $R^1(\bar{N}_1) = \sum_n R^1(C'_n)$.[19] Hence, the basis of the 1-cycles of $\bar{N}_1$ may be considered a set of 1-cycles each of which is contained in some C'_n. Since $S_1 \subset N_1$, it follows that $\bar{S}_1 \subset \bar{N}_1$ and that any 1-cycle of $\bar{S}_1$ is a 1-cycle of $\bar{N}_1$. There-

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[14] L. Zippin, *Transactions of American Mathematical Society*, vol. 34 (1932), pp. 705-721.

[15] See footnote 11, theorem (1. 3), p. 295.

[16] See footnote 14, theorem 5. 1, p. 709.

[17] L. Vietoris, *Mathematische Annalen*, vol. 97 (1927), theorem 8b, p. 470.

[18] See footnote 10, theorem 11, p. 177.

[19] See footnote 11, theorem 2. 5, p. 138.

fore, the basis of 1-cycles of $\bar{S}_1$ can be chosen so that it is a set of 1-cycles, each of which is contained in some C'_n and hence in some L'_n. It follows that $R^1(\bar{S}_1) = R^1(\sum_n L'_n) = 0$ because $R^1(L'_n) = 0$ for every n.

Furthermore, any component $\bar{S}_j$ of $\overline{\sum_i L_i}$ satisfies the same conditions as one of the sets $\bar{S}_1$ just considered. To see this we need to consider $\overline{\sum_i L_i} - \sum_i L_i$. Let x be any point of this set. Then x is a limit point of a subsequence $\{L_{i_j}\}$ of $\sum_i L_i$ and hence is contained in $\bar{S}_j = \overline{\sum L_{i_j}}$. Therefore, we conclude that $\overline{\sum_i L_i} = \sum_j \bar{S}_j$, where $\bar{S}_j$ is a continuum, $R^1(\bar{S}_j) = 0$, and $\bar{S}_i \cdot \bar{S}_j = 0$, $i \neq j$. Thus, the sets $\bar{S}_j$ satisfy the hypothesis of Lemma 3 and are contained in a continuum S of A such that $R^1(S) = 0$.

We show now that $C_i \cdot C_j = p$, a single point, for $i \neq j$. Define $T(A)$ as follows: $T(S)$ is a single point, T is a homeomorphism on $A - S$. If S does not cut C_i, $T(C_i)$ is a true cyclic element of B. If S cuts C_i, let D_{ij} be a component of $C_i - C_i \cdot S$. It will be demonstrated that $T(D_{ij} + C_i \cdot S) = B_{ij}$ is a true cyclic element of B. Since T is continuous, B_{ij} is a continuum. Also, B_{ij} can fail to be locally connected only at the point $T(S)$ and consequently is locally connected.[20] It remains to be seen that B_{ij} is cyclicly connected.

Let x and y represent any two points of D_{ij}. Since C_i is cyclicly connected and $D_{ij} \subset C_i$, the points x and y lie on some simple closed curve C' of C_i. If $C' \subset D_{ij}$, $T(C')$ is a simple closed curve in B_{ij} because T is a homeomorphism on D_{ij}. If $C' \not\subset D_{ij}$, there are two cases to be considered. The curve C' may satisfy the condition that one of the arcs xy of C' is in D_{ij}. Then there is an arc $uxyz$ of C' such that $\langle uz \rangle \subset D_{ij}$ and $u + z \subset C_i \cdot S$. Since $T(u) = T(z)$ and T is a homeomorphism on $\langle uz \rangle$, $T(uxyz)$ is a simple closed curve of B_{ij}. Hence, in this case $T(x)$ and $T(y)$ are on a simple closed curve in B_{ij}.

In the second case neither arc xy of C' is contained in D_{ij}. Then there are arcs uxv and wyz of C' such that $\langle uxv \rangle$ and $\langle wyz \rangle$ are in D_{ij} and u, v, w, z are points of S. There is an arc δ in D_{ij} joining x and y because D_{ij} is a component of $C_i - C_i \cdot S$. In the order x to y let s be the last point of uxv and t the first point of wyz on δ. There are four cases depending on the location of s and t. Suppose that s is a point of xv and t is a point of wy. Then the arc $\alpha = uxs + st + tyz$, where $st \subset \delta$ and $uxs + tys \subset C'$, satisfies the conditions that $x + y \subset \alpha$, $\langle \alpha \rangle \subset D_{ij}$, and $u + v \subset S$. Hence, $T(\alpha)$ is a simple closed curve of B_{ij} for the same reasons as those given in the preceding

[20] See footnote 7, theorem 8, p. 95.

case for $T(uxyz)$. The other cases corresponding to other positions of s and t on uxv and wyz can be discussed similarly. Therefore, we conclude that B_{ij} is cyclicly connected.

Since D_{ij} was any component of $C_i - C_i \cdot S$, it follows that $T(C_i)$ is a set of true cyclic elements of B having only the point $T(S)$ in common. Furthermore, B_i, any true cyclic element of B, is contained in $T(C_j)$ for some j. Hence, $B_i \cdot B_j = T(S)$ for $i \neq j$ and B is homeomorphic to A only if $C_i \cdot C_j$ is a single point p, $i \neq j$.

It follows from Theorem 5 that $A \equiv \sum_i C_i$ for if some C_i contains a cut point p' of A, $p' \neq p$, the closed extension of every component of $A - C_i$ contains a true cyclic element of A. Hence, there are in A true cyclic elements C_j and C_k such that $C_j \cdot C_i = p$ and $C_k \cdot C_i = p'$. It has just been shown that this is impossible since $C_i \cdot C_j = p$ for every $i \neq j$.

If some C_i, say C_1, is not a simple closed curve, let M' be a manifold homeomorphic to M and let $H(C_1)$ be the homeomorph of C_1 in M'. Then $H(C_1)$ is a cyclic connected Peanian continuum contained in a compact 2-manifold without boundary and hence Lemma 3 can be applied. Since the 2-cell E, which was used in the proof of Lemma 3, can be chosen so that $\bar{E}$ does not contain a certain point $p \neq x$, there is an acyclic cut set A_1 of $H(C_1)$ which does not contain $H(p)$. Define $T(A)$ thus: $T[H^{-1}(A_1)]$ is a single point, T is a homeomorphism on $A - A_1$. Then B contains two cut points, $T(p)$ and $T[H^{-1}(A_1)]$, and hence is not homeomorphic to A. We conclude that each C_i is a simple closed curve.

The Peanian continuum $A \equiv \sum_i C_i$, $C_i \cdot C_j = p$ for $i \neq j$, C_i a simple closed curve for every i, satisfies the hypothesis that B is homeomorphic to A for every T. Since T is continuous and monotonic, $T^{-1}(b)$ for every point b of B is a continuum. Moreover, $T^{-1}(b) \cdot C_i$, for any i, is a continuum because the intersection of any continuum and a true cyclic element of a continuous curve is a continuum. Consequently, T is monotonic on each simple closed curve C_i and $T(C_i)$ is a simple closed curve by the Corollary to Theorem 1. Also, $T(C_i) \cdot T(C_j) = T(p)$, $i \neq j$, because T is a $R_0{}^0$-transformation. Hence, B is homeomorphic to A.

COROLLARY. *If A contains only a finite number of true cyclic elements C_i, $i = 1, \cdots, n$, and $B = T(A)$ is homeomorphic to A for every T, then $A = \sum_{i=1}^{n} C_i$, $C_i \cdot C_j = p$, $i \neq j$, C_i is a simple closed curve.*

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